Quasihomomorphisms from the integers into matrices

Jan Draisma, Rob Eggermont, Tim Seynnaeve, Nafie Tairi, **Alejandro Vargas**, Emanuele Ventura

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Question Let $c \in \mathbb{Z}_{>0}$. Does there exist a constant $C = C(c)$ such that the following holds: For all $n \in \mathbb{Z}_{>0}$ and all functions $f : \mathbb{Z} \to \mathbb{C}^{n\times n}$ such that

$$\forall x, y \in \mathbb{Z}: \quad \mathrm{rk}(f(x+y) - f(x) - f(y)) \leq c, \quad (1)$$

there exists a matrix v such that

$$\forall x \in \mathbb{Z}: \quad \mathrm{rk}(f(x) - x \cdot v) \leq C? \quad (2)$$

Some remarks

- If $c = 0$ then f is a homomorphism of (additive) groups. Then $C = 0$.
- If f satisfies (1), we call f a c-**quasimorphism**.
- We focus on the space of diagonal matrices, which we identify with $\mathbb{C}^n$.
- The rank of a diagonal matrix is simply the **Hamming weight** w_H of the corresponding vector; i.e. the number of nonzero entries.
- We can without loss of generality assume that $v = f(1)$; this increases the constant C by a factor ≤ 2.

Example Take $c = 1$ and $n \geq 3$, and define

$$f : \mathbb{Z} \to \mathbb{Q}^n$$
$$x \mapsto \left(\left\lfloor\frac{2x+2}{5}\right\rfloor, \left\lfloor\frac{x+2}{5}\right\rfloor, \alpha_x, 0, \ldots, 0\right),$$
$$\text{where } \alpha_x = \begin{cases} 1 \text{ if } 5 \mid x, \\ 0 \text{ else.} \end{cases}$$

First couple of values:

$f(0) = (0, 0, 1, \ldots)$ $\quad f(8) = (3, 2, 0, \ldots)$
$f(1) = (0, 0, 0, \ldots)$ $\quad f(9) = (4, 2, 0, \ldots)$
$f(2) = (1, 0, 0, \ldots)$ $\quad f(10) = (4, 2, 1, \ldots)$
$f(3) = (1, 1, 0, \ldots)$ $\quad f(11) = (4, 2, 0, \ldots)$
$f(4) = (2, 1, 0, \ldots)$ $\quad f(12) = (5, 2, 0, \ldots)$
$f(5) = (2, 1, 1, \ldots)$ $\quad f(13) = (5, 3, 0, \ldots)$
$f(6) = (2, 1, 0, \ldots)$ $\quad f(14) = (6, 3, 0, \ldots)$
$f(7) = (3, 1, 0, \ldots)$ $\quad f(15) = (6, 3, 1, \ldots)$

- f is a 1-quasimorphism. For instance
$$f(14) - f(6) - f(8) = (1, 0, 0, \ldots)$$
has Hamming weight 1.
- $w_H(f(x) - x \cdot f(1)) \leq 3$, where equality is sometimes achieved.
- For $v = (\frac{2}{5}, \frac{1}{5}, 0, \ldots)$, it holds that
$$w_H(f(x) - x \cdot v) \leq 2 \qquad \forall x \in \mathbb{Z}.$$

c-quasimorphisms into diagonal matrices

Theorem 1. Let $c \in \mathbb{Z}_{\geq 0}$. There exists a constant $C = C(c) \in \mathbb{Z}_{\geq 0}$ such that for all $n \in \mathbb{Z}_{\geq 0}$ and all c-quasimorphisms $f : \mathbb{Z} \to \mathbb{Q}^n$, we have

$$\forall a \in \mathbb{Z}: \quad w_H(f(a) - a \cdot f(1)) \leq C.$$

Remarks:

- **Corollary:** Theorem 1 holds with $\mathbb{Q}$ replaced by any torsion-free abelian group (in particular: any field of characteristic 0), with the same $C(c)$.
- We can choose $C = 28c$; this is most likely not optimal.

Proofs

Write $f = (f_1, \ldots, f_n)$. We fix $a \in \mathbb{N}$ and write $[a] = \{1, \ldots, a\}$.

Define the *problem sets*

- $P_1(f_i) := \{x \in [a] \mid f_i(x+1) \neq f_i(x) + f_i(1)\}$;
- $P_a(f_i) := \{x \in [a] \mid f_i(x+a) \neq f_i(x) + f_i(a)\}$;
- $P(f_i) := \{(x, y) \in [a] \times [a] \mid f_i(x+y) \neq f_i(x) + f_i(y)\}$.

Claim 1: Let $g : [2a] \to \mathbb{Q}$ be any map such that $g(a) \neq ag(1)$, then

$$|P_1(g)| \geq qa \quad \text{or} \quad |P_a(g)| \geq pa \quad \text{or} \quad |P(g)| \geq ra^2,$$

where $q = 0.1167$, $p = 0.165$, and $r = 0.0765$.

Why Claim 1 implies the theorem:

$$w_H(f(a) - a \cdot f(1)) > C$$

$$\overset{\text{Claim 1}}{\Longrightarrow} \text{WLOG} \quad \#\{i : |P(f_i)| \geq ra^2\} > \frac{C}{3}$$

$$\Longrightarrow \exists (x, y) \in [a] \times [a] \text{ such that } \#\{i : (x, y) \in P(f_i)\} > c$$

$$\Longrightarrow f \text{ is not a } c\text{-quasimorphism.}$$

Why you should believe Claim 1:

Fact: If $g : \mathbb{Z}/a\mathbb{Z} \to \mathbb{Q}$ is a group morphism, then $g \equiv 0$.

- WLOG $g(a) = 0$ and $g(1) \neq 0$.
- Observe:
  - $P_a(g)$ small means: "g is almost a map from $\mathbb{Z}/a\mathbb{Z}$."
  - $P(g)$ small means: "g is almost a group homomorphism."
  - If g is close to being constant, then $P_1(g)$ is large.
- So we are done if we can make the following precise:

If g is **almost** a group morphism $\mathbb{Z}/a\mathbb{Z} \to \mathbb{Q}$, then **almost** $g \equiv 0$. Want to know how? See [1].

1-quasimorphisms into symmetric matrices

Theorem 2 If $f : \mathbb{Z} \to \mathrm{Sym}(n \times n, \mathbb{Q})$ is a 1-quasimorphism, there is an $A \in \mathrm{Sym}(n \times n, \mathbb{Q})$ such that

$$\mathrm{rk}(f(x) - x \cdot A) \leq 2 \quad \forall x \in \mathbb{Z}.$$

Remarks

- In particular, for $c = 1$ the bound $C = 28$ from Theorem 1 can be improved to $C = 2$.

Proof

- WLOG can assume that $f(1) = 0$.
- Then we find that $\mathrm{rk}(f(x+1) - f(x)) \leq 1$.
- So we write $\Delta_f(x) = f(x+1) - f(x)$. This is a sequence of rank one matrices. Note that $f(x) = \Delta(1) + \cdots + \Delta(x-1)$ for $x > 0$.
- For instance, in the example our sequence looks like Table 1 below.
- If f is a 1-quasimorphism, then
$$\mathrm{rk}(\Delta_f(1) + \cdots + \Delta_f(k)$$
$$- \Delta_f(x) - \cdots - \Delta_f(x+k)) = 0.$$
- With some work, we can show that then Δ_f must look as follows Table 2 below.

  where $ab \cdots ba$ is a length $p-2$ palindromic sequence of matrices that lie in a fixed $\mathbb{C}^2 \otimes \mathbb{C}^2$.
- Then we can take $A = \frac{f(p-1)}{p} = \frac{a+b+\cdots+b+a}{p}$. Indeed:
  - If $x = kp$, then $f(x) = k \cdot (a + b + \cdots + b + a) + \gamma = kpA + \gamma$, so $\mathrm{rk}(f(x) - x \cdot A) = \mathrm{rk}(\gamma) \leq 1$.
  - Else $p \nmid x$, and then both $f(x)$ and A are in the aforementioned $\mathbb{C}^2 \otimes \mathbb{C}^2$, which implies $\mathrm{rk}(f(x) - x \cdot A) \leq 2$.

Conclusions

We answered Question 1 for diagonal matrices, and for symmetric matrices if $c = 1$. We don't know a proof for general matrices (even if $c = 1$); or for symmetric matrices and $c > 1$.

References

[1] J. Draisma, R. Eggermont, T. Seynnaeve, N. Tairi, E. Ventura
Quasihomomorphisms from the integers into Hamming metrics
ArXiv: 2204.08392

[2] T. Seynnaeve, N. Tairi, **A. Vargas**
One-quasihomomorphisms from the integers into symmetric matrices ArXiv: 2302.01611

Table 1:

x	$\cdots$	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
$\Delta(x)$	$\cdots$	e_1	e_2	e_1	e_3	$-e_3$	e_1	e_2	e_1	e_3	$-e_3$	e_1	e_2	e_1	e_3	$-e_3$	e_1	e_2	e_1	$\cdots$

Table 2:

x	$\cdots$			$\cdots$		-2	-1	0	1		$\cdots$				p			$\cdots$				$2p$	$\cdots$
$\Delta(x)$	$\cdots$	a	b	$\cdots$	b	a	α	$-\alpha$	a	b	$\cdots$	b	a	β	$-\beta$	a	b	$\cdots$	b	a	γ	$-\gamma$	$\cdots$