Counting tropical morphisms from metric graphs to metric trees

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(slides available at https://vargas.page/)

Overall goal: solve an enumerative problem in tropical geometry modelled after a classical problem

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Motivation: tropical geometry deals with piecewise linear objects that arise as limits of degenerations on classical algebraic varieties

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Motivation: tropical geometry deals with piecewise linear objects that arise as limits of degenerations on classical algebraic varieties

Thus, a central question is what information survives this degeneration, hence tropical results that show a strong analogy to the classical setting are interesting.

Enumerative geometry

Enumerative geometry

A question's recipe

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(A) Some geometrical objects

Enumerative geometry

A question's recipe

Ⓐ Some geometrical objects

(e.g. lines, curves, maps, etc.)

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Ⓑ Some conditions

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mix them to get

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Examples:

Enumerative geometry

A question's recipe
(A) Some geometrical objects
(e.g. lines, curves, maps, etc.)
(B) Some conditions
mix them to get
(?) How many in (A) satisfy (B) ?

Examples:
[1] Take:

Enumerative geometry

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Ⓐ Some geometrical objects

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Examples:

[1] Take:

Ⓐ $=$ curves that are:

Enumerative geometry

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Examples:
[1] Take:
Ⓐ $=$ curves that are:
irreducible

Enumerative geometry

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Examples:

[1] Take:

Ⓐ = curves that are:

irreducible

planar

Enumerative geometry

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Examples:

[1] Take:

Ⓐ = curves that are:

irreducible

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complex

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Examples:

[1] Take:

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planar

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sing. are nodal

Enumerative geometry

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Examples:

1 Take:

Ⓐ = curves that are:

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sing. are nodal

degree d

Enumerative geometry

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Examples:

$\boxed{1}$ Take:

Ⓐ = curves that are:

- irreducible
- planar
- complex
- sing. are nodal
- degree d
- genus g

Enumerative geometry

A question's recipe

(A) Some geometrical objects

(e.g. lines, curves, maps, etc.)

(B) Some conditions

mix them to get

(?) How many in (A) satisfy (B) ?

Examples:

[1] Take:

(A) = curves that are:

irreducible

planar

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genus g

We have

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Ⓐ Some geometrical objects
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Examples:
[1] Take:
Ⓐ = curves that are:
- irreducible
- planar
- complex
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We have

$$\dim Ⓐ = 3d - 1 + g,$$

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Ⓑ = pass through fixed points

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$$\mathcal{P} = \{p_1, \ldots, p_{3d-1+g}\}$$

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Rem: independent of $\mathcal{P}$

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$\mathcal{P} = \{p_1, \ldots, p_{3d-1+g}\}$
Ⓘ = Finite answer, denoted:
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Rem: independent of $\mathcal{P}$
(by intersection theory)

Enumerative geometry - Examples

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Examples:

$\boxed{1}$ $N(d, g) =$ degree-d genus-g complex plane curves through $3d - 1 + g$ points

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Examples:

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2 Take:

Enumerative geometry - Examples

A question's recipe

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Examples:

[1] $N(d, g) =$ degree-d genus-g complex plane curves through $3d - 1 + g$ points

[2] Take:

Ⓐ = non-constant rational maps $f : X \to \mathbb{P}^1$

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X is a curve that is

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2 Take:

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X is a curve that is smooth

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[1] $N(d, g) =$ degree-d genus-g complex plane curves through $3d - 1 + g$ points

[2] Take:

Ⓐ $=$ non-constant rational maps $f : X \to \mathbb{P}^1$

X is a curve that is
- smooth
- has even genus $2g'$

Enumerative geometry - Examples

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X is a curve that is
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has even genus $2g'$

We have

Enumerative geometry - Examples

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X is a curve that is

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We have

$\dim Ⓐ = 2 \deg f - 2g' - 2$

Enumerative geometry - Examples

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We have
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Ⓑ = Require:

Enumerative geometry - Examples

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Examples:

1 $N(d, g)$ = degree-d genus-g complex plane curves through $3d - 1 + g$ points

2 Take:

Ⓐ = non-constant rational maps $f : X \to \mathbb{P}^1$

X is a curve that is
smooth
has even genus $2g'$

We have

$$\dim Ⓐ = 2 \deg f - 2g' - 2$$

Ⓑ = Require:
$\deg f = g' + 1$

Enumerative geometry - Examples

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Examples:

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2 Take:

Ⓐ = non-constant rational maps
$f : X \to \mathbb{P}^1$
X is a curve that is
smooth
has even genus $2g'$

We have
$$\dim Ⓐ = 2 \deg f - 2g' - 2$$

Ⓑ = Require:
$\deg f = g' + 1$

? = number we call $C(g')$

Enumerative geometry - Examples

A question's recipe

Ⓐ Some geometrical objects (e.g. lines, curves, maps, etc.)

Ⓑ Some conditions

mix them to get

ⓘ How many in Ⓐ satisfy Ⓑ ?

Examples:

$\boxed{1}$ $N(d, g) =$ degree-d genus-g complex plane curves through $3d - 1 + g$ points

$\boxed{2}$ Take:

Ⓐ = non-constant rational maps
$f : X \to \mathbb{P}^1$
X is a curve that is
smooth
has even genus $2g'$

We have
$\dim Ⓐ = 2 \deg f - 2g' - 2$

Ⓑ = Require:
$\deg f = g' + 1$

ⓘ = number we call $C(g')$

Rem: independent of X

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Examples:

1 $N(d, g)$ = degree-d genus-g complex plane curves through $3d - 1 + g$ points

2 Take:

Ⓐ = non-constant rational maps
$f : X \to \mathbb{P}^1$
X is a curve that is
smooth
has even genus $2g'$

We have
$\dim Ⓐ = 2 \deg f - 2g' - 2$

Ⓑ = Require:
$\deg f = g' + 1$

? = number we call $C(g')$
Rem: independent of X
(again by intersection theory)

Enumerative geometry - Tropicalizing examples

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Ⓐ Some geometrical objects
(e.g. lines, curves, maps, etc.)
Ⓑ Some conditions
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Examples:
$\boxed{1}$ $N(d, g) =$ degree-d
genus-g complex plane curves
through $3d - 1 + g$ points

Enumerative geometry - Tropicalizing examples

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Examples:

$\boxed{1}$ $N(d, g) =$ degree-d genus-g complex plane curves through $3d - 1 + g$ points

$\boxed{2}$ $C(g') =$ degree-$(g' + 1)$ morphisms from genus-$2g'$ smooth complex curve to $\mathbb{P}^1$

Enumerative geometry - Tropicalizing examples

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Examples:

1 $N(d, g)$ = degree-d genus-g complex plane curves through $3d - 1 + g$ points

2 $C(g')$ = degree-$(g'+1)$ morphisms from genus-$2g'$ smooth complex curve to $\mathbb{P}^1$

Now we tropicalize the examples

Enumerative geometry - Tropicalizing examples

A question's recipe

(A) Some geometrical objects

(e.g. lines, curves, maps, etc.)

(B) Some conditions

mix them to get

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Examples:

[1] $N(d, g) =$ degree-d genus-g complex plane curves through $3d - 1 + g$ points

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Now we tropicalize the examples

[3] $N^{\text{trop}}(d, g) =$ degree-d genus-g tropical plane curves through $3d - 1 + g$ points $\mathcal{P}$

Enumerative geometry - Tropicalizing examples

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Examples:

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Now we tropicalize the examples

[3] $N^{\text{trop}}(d, g) =$ degree-d genus-g tropical plane curves through $3d - 1 + g$ points $\mathcal{P}$

Rem. independent of $\mathcal{P}$ by correspondence theorem.

Enumerative geometry - Tropicalizing examples

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Examples:

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Now we tropicalize the examples

[3] $N^{\text{trop}}(d, g) =$ degree-d genus-g tropical plane curves through $3d - 1 + g$ points $\mathcal{P}$
Rem. independent of $\mathcal{P}$ by correspondence theorem.
(Mikhalkin 05)

Enumerative geometry - Tropicalizing examples

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Examples:

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Now we tropicalize the examples

3 $N^{\text{trop}}(d, g) =$ degree-d genus-g tropical plane curves through $3d - 1 + g$ points $\mathcal{P}$
Rem. independent of $\mathcal{P}$ by correspondence theorem. (Mikhalkin 05)

4 $C^{\text{trop}}(g') =$ degree-$(g' + 1)$ tropical morphisms from genus-$2g'$ tropical curve Γ to metric trees

Enumerative geometry - Tropicalizing examples

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(e.g. lines, curves, maps, etc.)
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Examples:

[1] $N(d,g) =$ degree-d genus-g complex plane curves through $3d-1+g$ points

[2] $C(g') =$ degree-$(g'+1)$ morphisms from genus-$2g'$ smooth complex curve to $\mathbb{P}^1$

Now we tropicalize the examples

[3] $N^{\text{trop}}(d,g) =$ degree-d genus-g tropical plane curves through $3d-1+g$ points $\mathcal{P}$
Rem. independent of $\mathcal{P}$ by correspondence theorem. (Mikhalkin 05)

[4] $C^{\text{trop}}(g') =$ degree-$(g'+1)$ tropical morphisms from genus-$2g'$ tropical curve Γ to metric trees
Rem. independent of Γ by combinatorics/balancing condition (Vargas 22)

Counting tropical morphisms

Ⓐ Some geometrical objects
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Ⓐ for $\boxed{4}$:
(abstract) tropical curve:

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Ⓐ for $\boxed{4}$:
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finite graph $G = (V, E)$

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$C^{\text{trop}}(g') =$ degree-$(g'+1)$
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Ⓐ for $\boxed{4}$ **:**
(abstract) tropical curve:
Is a pair of:

finite graph $G = (V, E)$
and length function
$\ell : E(G) \to \mathbb{R}_{\geq 0}$

Counting tropical morphisms

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Ⓐ for $\boxed{4}$:
(abstract) tropical curve:
Is a pair of:
finite graph $G = (V, E)$
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from $(G, \ell) \to$ get a compact
metric space $|\Gamma|$

Counting tropical morphisms

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Ⓐ for $\boxed{4}$ **:**
(abstract) tropical curve:
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metric space $|\Gamma|$
glue together intervals

Counting tropical morphisms

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$C^{\text{trop}}(g') = \text{degree-}(g'+1)$
tropical morphisms from
genus-$2g'$ tropical curve Γ to
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Ⓐ for $\boxed{4}$:
(abstract) tropical curve:
Is a pair of:
finite graph $G = (V, E)$
and length function
$\ell : E(G) \to \mathbb{R}_{\geq 0}$

from $(G, \ell) \to$ get a compact
metric space $|\Gamma|$
glue together intervals
$\bigsqcup_{e \in E(G)} [0, \ell(e)] / \sim$

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$\bigsqcup_{e \in E(G)} [0, \ell(e)] / \sim$
where the gluing is
$\sim = e \leftarrow A \rightarrow e'$ gets glued

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from $(G, \ell) \to$ get a compact
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$\bigsqcup_{e \in E(G)} [0, \ell(e)] / \sim$
where the gluing is
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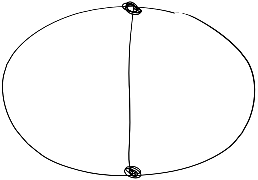
Example:



Counting tropical morphisms

Ⓐ Some geometrical objects
Ⓑ Some conditions
⑦ How many in Ⓐ satisfy Ⓑ ?
$C^{\text{trop}}(g') =$ degree-$(g'+1)$
tropical morphisms from
genus-$2g'$ tropical curve Γ to
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Ⓐ for $\boxed{4}$ **:**
(abstract) tropical curve:
Is a pair of:
  finite graph G
  and length function
    $\ell : E(G) \to \mathbb{R}_{\geq 0}$

from $(G, \ell) \to$ get a compact
topological realization
glue together intervals
  $\bigsqcup_{e \in E(G)} [0, \ell(e)] / \sim$
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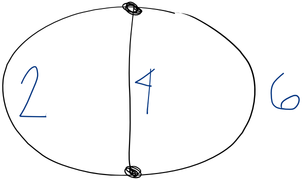

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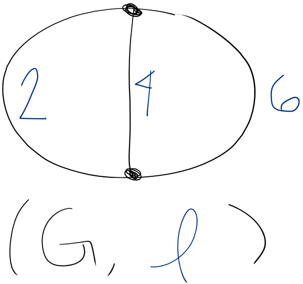

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well-defined local degree $m_\varphi(x)$

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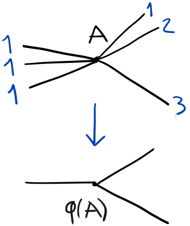

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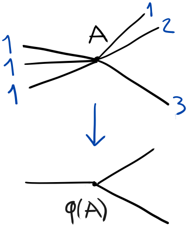

- Effective ramification divisor

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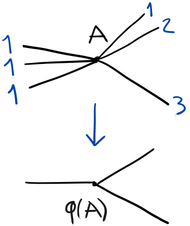

• Effective ramification divisor
rem: is a realizability condition
• **degree of** Φ: sum of all local degrees over any fiber of $p \in \Delta$

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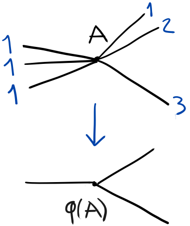

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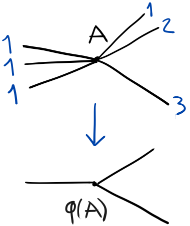

• Effective ramification divisor
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• **tree** genus-0 metric graph (there are no cycles)

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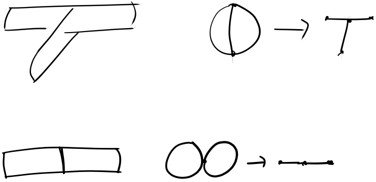

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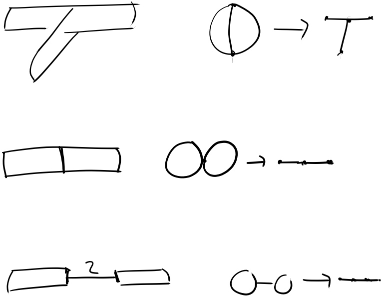

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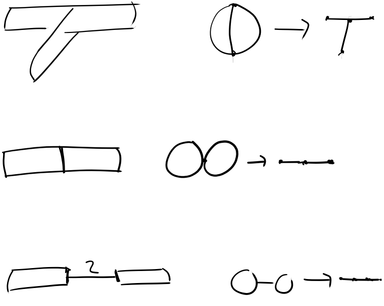

Rem: middle example obtained from other two

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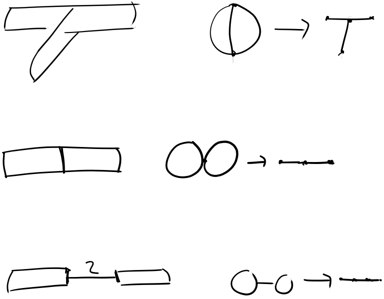

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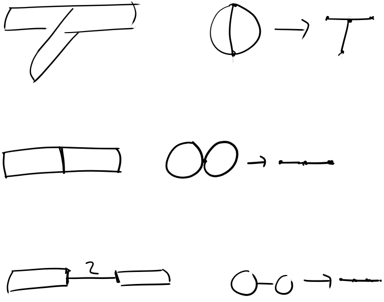

Rem: middle example obtained
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suggests how to glue families

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Take a "general" (i.e. trivalent) graph G

Counting tropical morphisms

Ⓐ Some geometrical objects
Ⓑ Some conditions
? How many in Ⓐ satisfy Ⓑ ?

$C^{\text{trop}}(g') =$ degree-$(g'+1)$ tropical morphisms from genus-$2g'$ tropical curve to metric trees

Ⓐ for $\boxed{4}$:

(abstract) tropical curve:
$\Gamma = (G, \ell : E(G) \to \mathbb{R}_{\geq 0})$

genus:
$g(\Gamma) = \#E(G) - \#V(G) + 1$

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dim $C_G = 3g - 3$

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Calculate
dim $C_G = 3g - 3$
dim $C_\varphi = 2g + 2d - 5$

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Finite answer if

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Finite answer if

$$\dim C_G = \dim C_\varphi$$

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Thus, $\deg \varphi = g' + 1$

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Finite answer if
$\dim C_G = \dim C_\varphi$
Thus, $\deg \varphi = g' + 1$

? What is $C^{\text{trop}}(g')$?

Enumerative geometry - A solution

A recipe for solution

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A recipe for solution

follow three steps:

Enumerative geometry - A solution

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follow three steps:

Ⓘ Put a geometry on Ⓐ

Enumerative geometry - A solution

A recipe for solution

follow three steps:

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(II) Consider locus of solution

and deform it.

Enumerative geometry - A solution

A recipe for solution

follow three steps:

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(II) Consider locus of solution and deform it.

(III) Find a favorable situation to do the count.

Enumerative geometry - A solution

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- C_G is naively easy

- but, account for isometry:

Enumerative geometry - A solution

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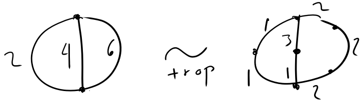

Enumerative geometry - A solution

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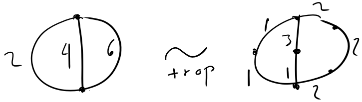

so define,

Enumerative geometry - A solution

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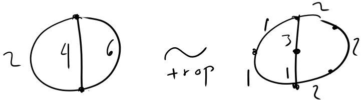

so define,

$TropModel(G) =$ "minimal model"

Enumerative geometry - A solution

A recipe for solution

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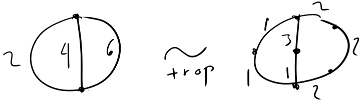

so define,

TropModel(G) = "minimal model"

(whatever minimal is here)

Enumerative geometry - A solution

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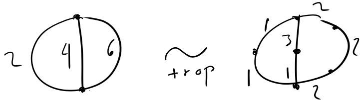

so define,
TropModel(G) = "minimal model"
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rem: no valency-2 vertices

Geometry for C_φ

A recipe for solution

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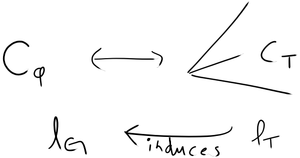

but we want to regard C_φ in

Geometry for C_φ

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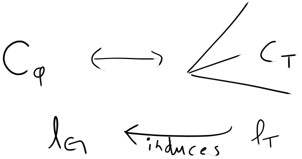

but we want to regard C_φ in

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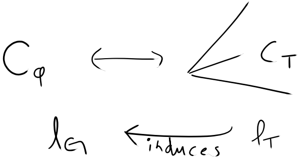

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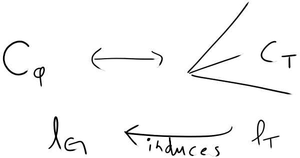

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$$A_\varphi : C_\varphi \to C_{TropModel(G)}$$

Geometry for C_φ

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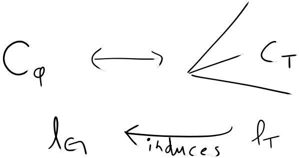

but we want to regard C_φ in

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is linear.

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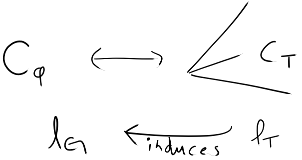

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only consider φ s.t. A_φ is injective

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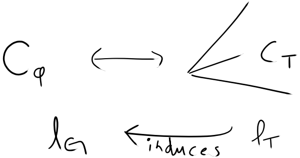

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Rem. count for (?) is right for generic Γ

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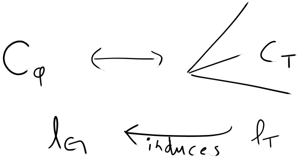

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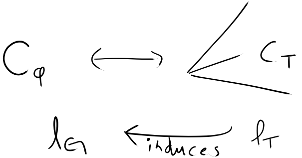

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Rem. count for ? is right for generic Γ

and lower bounds count for special Γ

(as in classical brill noether theory)

Deforming tropical morphisms

A recipe for solution

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Deforming tropical morphisms

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Idea is to lift deformation paths

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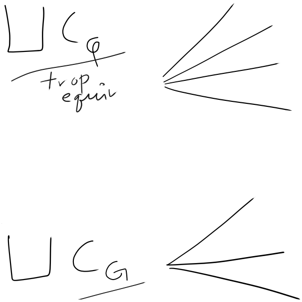

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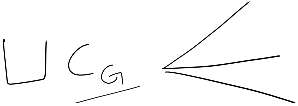

draw path in space below

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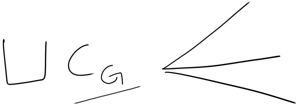

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(III) Count in a favorable situation

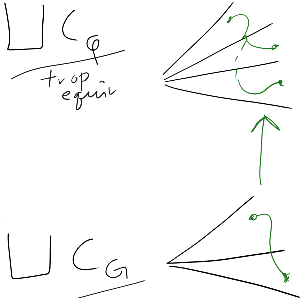
$C^{\text{trop}}(g') =$ degree-$(g'+1)$ tropical morphisms from genus-$2g'$ tropical curve to metric trees

(I) for $\boxed{4}$:

Put a geometry on C_G and C_φ

(II) for $\boxed{4}$:

Idea is to lift deformation paths



draw path in space below

lift using that A_φ is injective

Deforming tropical morphisms

A recipe for solution

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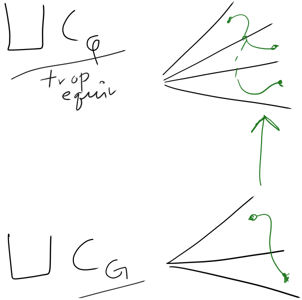

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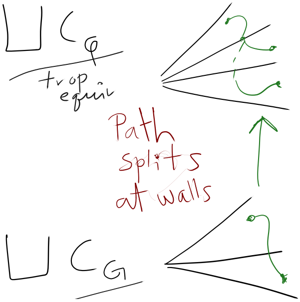

For the combinatorics at walls: https://youtu.be/28zHtR1Kr4Y?t=40

Caterpillars of loops

A recipe for solution

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(II) Deform locus of solution

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Caterpillars of loops

(III) for $\boxed{4}$: find "easy" graph

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- caterpillar of loops

Caterpillars of loops

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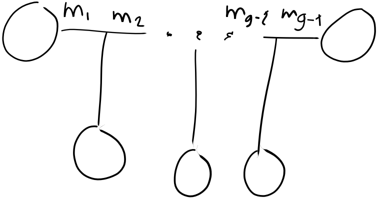

Caterpillars of loops

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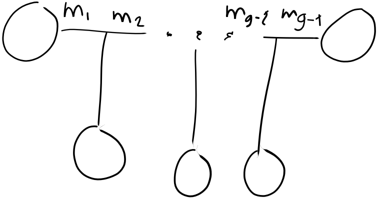

trop. morph. determined by

Caterpillars of loops

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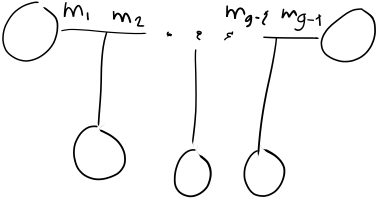
- caterpillar of loops



trop. morph. determined by $m_1, m_2, \ldots, m_{g-2}, m_{g-1}$

Caterpillars of loops

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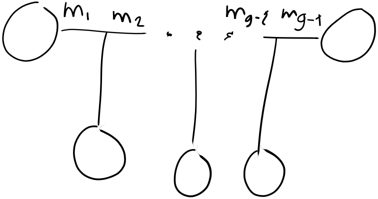
- caterpillar of loops



trop. morph. determined by $m_1, m_2, \ldots, m_{g-2}, m_{g-1}$ of the inner bridges

Caterpillars of loops

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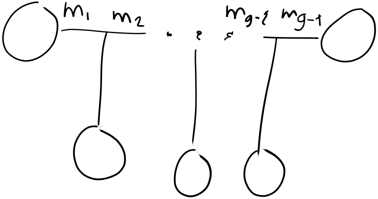
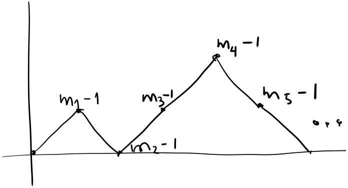
- caterpillar of loops



trop. morph. determined by $m_1, m_2, \ldots, m_{g-2}, m_{g-1}$ of the inner bridges and correspond to Dyck paths:



Caterpillars of loops

A recipe for solution

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Calculate for caterpillar of loops

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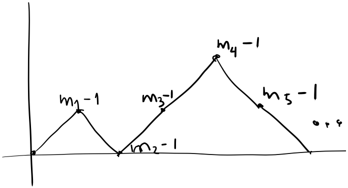

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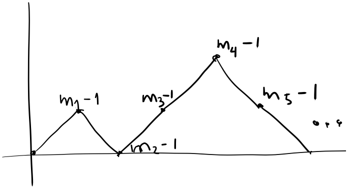

Is a lattice path that

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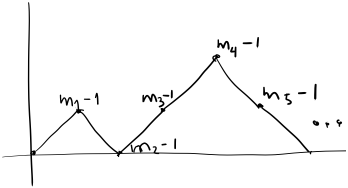

Is a lattice path that

- starts and ends at $y = 0$

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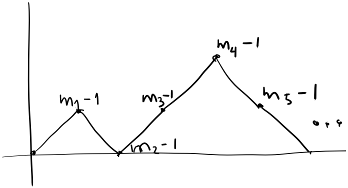

Is a lattice path that

- starts and ends at $y = 0$
- each step: up/down by 1

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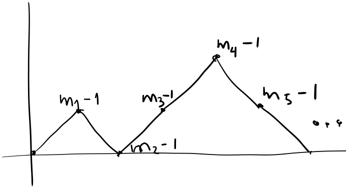

Is a lattice path that

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- never below x-axis

Caterpillars of loops

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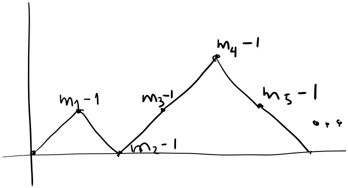

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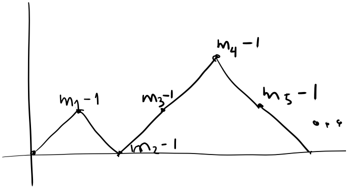

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$$= \frac{1}{g'+1}\binom{2g'}{g'}$$

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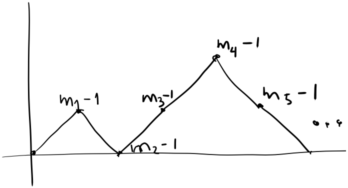

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so-called catalan number,

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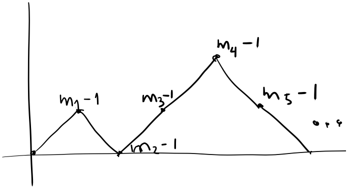

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so-called catalan number,
as in the classical setting $C(g')$

Caterpillars of loops

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Remarks:

Caterpillars of loops

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Remarks:

• tropical equivalence also disregards 1-valent vertices

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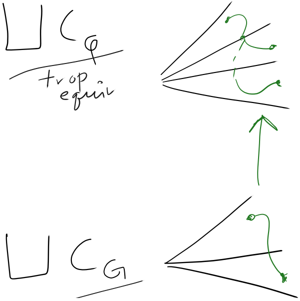
Calculate for caterpillar of loops

Remarks:

- tropical equivalence also disregards 1-valent vertices
- the map



is similar to $\Phi : \Gamma \to \Delta$

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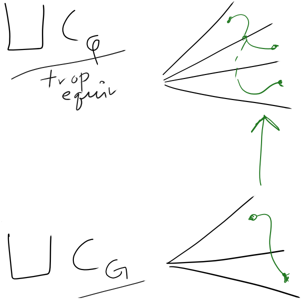

is similar to $\Phi : \Gamma \to \Delta$

there is a category (indexed branched covers) containing both maps.