A story of tropical morphisms and moduli spaces

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This non-technical summary discusses with informal metaphors the intuitive ideas behind the objects and the goals of my phd thesis.

Moduli spaces. Consider the phonebook of a city. Here are two reasons why this physical book is useful:

- Each person has exactly one entry in the phonebook (is a bijection).
- There is an order that makes searching for an entry easy (has a topology).

Consider the following two questions:

(A) Can we compare the population of two cities?
(B) Is the last name of a given person common?

Question (A) is a *global question*, concerning the totality of the object. Since the phonebook is a bijection, the population of a city is proportional to the length of the phonebook. So one just needs to visually inspect which phonebook is bigger. Question (B) is a *local question*, concerning a part of the object close to a point of interest. Assuming alphabetical order, first we locate the last name, and then check if the topological neighbourhood is big: count how many entries before, and after, of the chosen person have the same last name.

A *moduli space* is a phonebook for geometrical objects, useful to solve both global and local questions. Making a phonebook is quite a laborious process; so is constructing a moduli space. It is also very rewarding, hence the study of moduli spaces has been at the forefront of mathematics for the past 150 years.

Tropical geometry. Consider your favourite dinosaur. Our knowledge of it is indirect: we haven't seen it in a natural habitat, but instead have studied fossils. The *deformation process* that transformed the dinosaur into its skeleton lost a great deal of information (e.g. colour, body weight, whether it had feathers). Yet, enough is *preserved* to have paleontology as a field of science.

Tropical geometry is paleontology for mathematical dinosaurs called *algebraic varieties*. These are geometric objects described by polynomial equations, e.g. a circle in the plane is described by $x^2 + y^2 = 1$. The skeletons are *polyhedral complexes*. Here we can picture the shape of a quartz crystal, something with straight edges, straight surfaces, etc. See Figure 1. Tropical geometry is a new field at the intersection of algebraic geometry and combinatorics, with a great development in the last 20 years. Its name honours one of its founding fathers, Brazilian mathematician Imre Simon.

What do we gain from this? Think about how in paleontology skeletons are easier to manage than living creatures. In tropical geometry it is easier to manage polyhedral complexes than algebraic varieties, because the study is mostly combinatorial. Besides studying the skeletons, tropical geometry studies the processes that deform an algebraic variety into a polyhedral complex. The point is to establish *correspondence theorems* that tell us which information is retained, and to *develop efficient methods to compute* polyhedral complexes.

Coverings. Consider packing a suitcase. All clothes must cover the same space in the suitcase. Clearly, bigger more sophisticated clothes have to be folded more times to fit. So the number of folds for a particular dress encodes, roughly, how complicated this particular dress is. Suppose we are given a folded dress and we are challenged to determine the number of folds without unfolding it. We can take scissors to do a cross-section cut, then count the number of layers to get the answer. This gives a rough idea of how complicated this dress used to be.

A *covering* is a list of three things: the dress, the suitcase, and the specific way the dress was folded. The *degree* of a covering is the number of layers in a cross-section cut. Given a dress and a suitcase, we ask the *folding problem*: What is the *best* way to fold the dress? In quasi-mathematical terms we would say *what is the minimum degree of a covering for a specific dress and suitcase.* This is a fairly difficult problem, driving ongoing research. The reason for this is that essentially we are expressing a complicated geometrical object in terms of a simpler one, a process conceptually akin to expressing a natural number as a product of prime numbers.

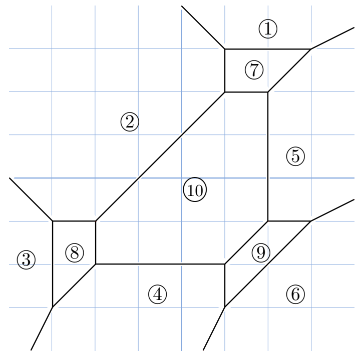

FIGURE 1. A tropical variety associated to the following optimization problem:

$$\max(-5+2y, -1-x+y, -5-2x, -2-y, -2+x,$$
$$-7+2x-2y, -2+y, -2-x, -3+x-y, 0)$$

The maximum is attained at $-5+2y$ in region ①, the maximum is attained at $-1-x+y$ in region ②, and so on.

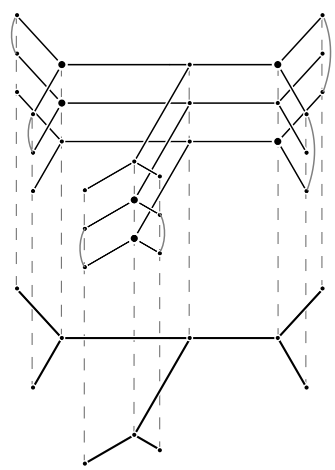

FIGURE 2. A degree-3 covering that folds the variety from Figure 1.

Results. If both clothes and suitcase are polyhedral complexes, then we call the covering a *tropical covering*. Do not panic, we still are folding things, just that now we fold the skeletons that tropical geometry cares about. And we still care about finding the most *efficient* way of folding. Efficient means *low-degree tropical coverings*.

This thesis constructs a moduli space, a phone book, for low-degree tropical coverings. Let us name this phonebook $\mathcal{G}^{\text{trop}}_{g\to 0,d}$. We have shown, in a beautiful result, that $\mathcal{G}^{\text{trop}}_{g\to 0,d}$ is a polyhedral complex, and that it can be folded and used to cover another polyhedral complex $\mathcal{M}^{\text{trop}}_{g}$ which is an important moduli space. Please pause a moment and appreciate the self-referential nature of the result: *a moduli space for tropical coverings is itself a tropical covering*. These *meta qualities* are common in moduli spaces and make them, in my opinion, aesthetically appealing. Another beautiful result is that we calculate the degree of the covering that folds $\mathcal{G}^{\text{trop}}_{g\to 0,d}$ and get a *Catalan number*. These numbers have profound combinatorial meanings, but *this note is too narrow to contain them*.

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